

4.6 Rational Functions and Their Graphs

▼ Definition of a Rational Function

A rational function is of the form

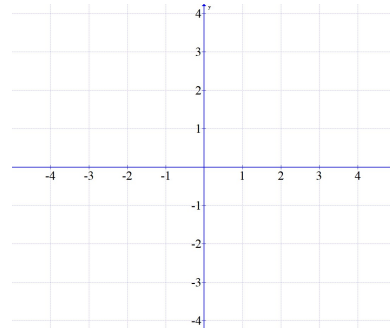
$$f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0}$$

where $p(x)$ and $q(x)$ are polynomial functions and $q(x)$ is not equal to zero.

▼ Examples: Graphing Basic Rational Functions by Plotting Points

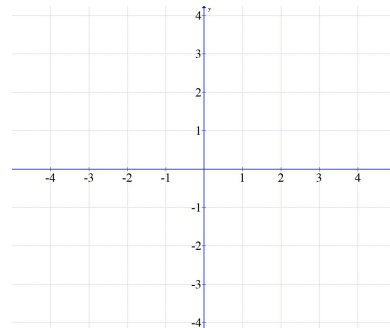
▼ Example 1: Graph $f(x) = \frac{1}{x}$

x	y	(x,y)



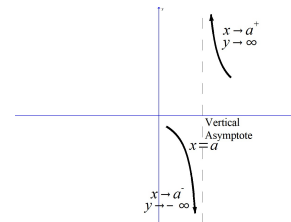
▼ Example 2: Graph $f(x) = \frac{1}{x^2}$

x	y	(x,y)



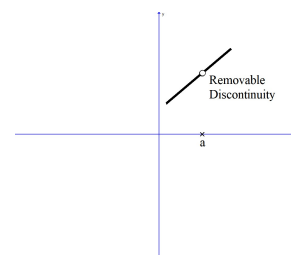
▼ Definition of a Vertical Asymptote

The line $x = a$ is a vertical asymptote of the graph of a function f if $f(x)$ increases or decreases without bound as x approaches a . Symbolically, if $f(x) \rightarrow \infty$ or $f(x) \rightarrow -\infty$ as $x \rightarrow a$ then $x = a$ is a vertical asymptote.



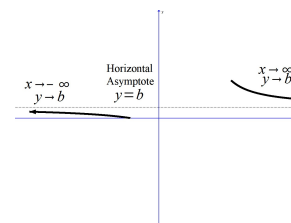
▼ Definition of a Removable Discontinuity

If $f(x) = \frac{p(x)}{q(x)}$ is a rational function in which $p(x)$ and $q(x)$ have common factors and a is a zero of the common factor, then $x = a$ is the location of a removable discontinuity (hole) of the graph of f .



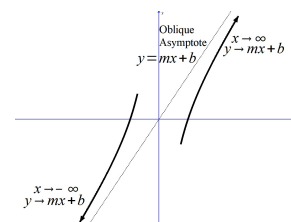
▼ Definition of a Horizontal Asymptote

The line $y = b$ is a horizontal asymptote of the graph of a function f if $f(x)$ approaches b as x increases or decreases without bound. Symbolically, if $f(x) \rightarrow b$ as $x \rightarrow \infty$ or $x \rightarrow -\infty$ then $y = b$ is a horizontal asymptote.



▼ Definition of an Oblique/Slant Asymptote

The line $y = ax + b$ is an oblique asymptote of the graph of a function f if $f(x)$ approaches $ax + b$ as x increases or decreases without bound. Symbolically, if $f(x) \rightarrow ax + b$ as $x \rightarrow \infty$ or $x \rightarrow -\infty$ then $y = ax + b$ is an oblique asymptote.

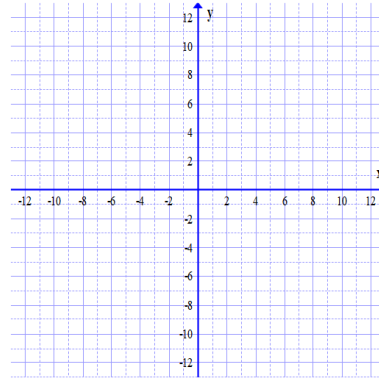


▼ Examples: Graphing Transformations of Basic Rational Functions

▼ Example 1: Graph the function using Transformations. $f(x) = -\frac{1}{x-2} + 5$

Base Function: _____

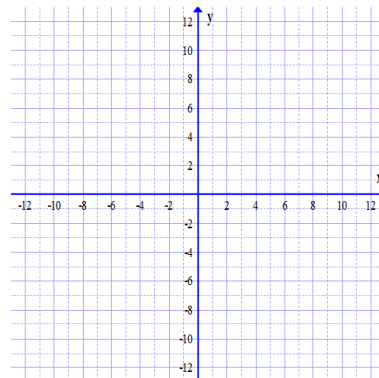
List Transformations:



▼ Example 2: Graph the function using Transformations. $f(x) = -\frac{1}{(x-2)^2} + 5$

Base Function: _____

List Transformations:



▼ Example 1-4: Analyze the Graph of a Rational Function

Domain:

x-int:

Vertical Asymptote:

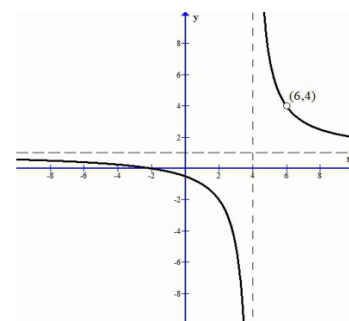
Horizontal Asymptote:

Range:

y-int:

Removable
Discontinuity:

Oblique Asymptote:



Domain:

x-int:

Vertical Asymptote:

Horizontal Asymptote:

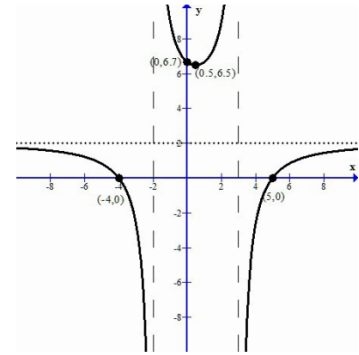
Range:

y-int:

Removable

Discontinuity:

Oblique Asymptote:



Domain:

x-int:

Vertical Asymptote:

Horizontal Asymptote:

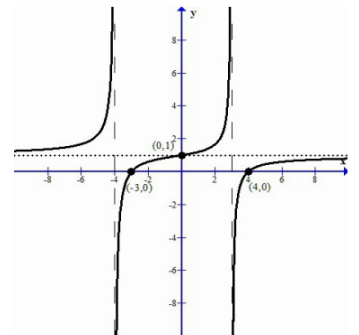
Range:

y-int:

Removable

Discontinuity:

Oblique Asymptote:



Domain:

x-int:

Vertical Asymptote:

Horizontal Asymptote:

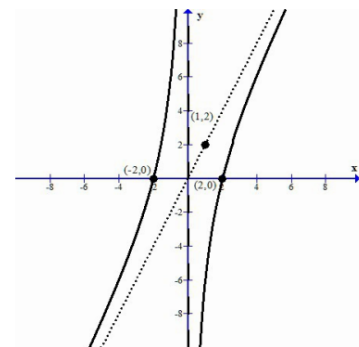
Range:

y-int:

Removable

Discontinuity:

Oblique Asymptote:



▼ Finding Vertical Asymptotes and Removable Discontinuities of a Rational Function

A rational function has a restricted value when the denominator is equal to zero because division by zero is undefined. The restricted value can produce a vertical asymptote or a removable discontinuity in the graph of the rational function.

Vertical Asymptote: If $f(x) = \frac{p(x)}{q(x)}$ is a rational function in which $p(x)$ and $q(x)$ have no common factors and a is a zero of the denominator (a is a solution to $q(x) = 0$), then $x = a$ is a vertical asymptote of the graph of f .

Removable Discontinuity: If $f(x) = \frac{p(x)}{q(x)}$ is a rational function in which $p(x)$ and $q(x)$ have common factors and a is a zero of the common factor, then $x = a$ is the location of a removable discontinuity (hole) of the graph of f .

▼ Finding Vertical Asymptotes and Removable Discontinuities

1. Factor the numerator and denominator.
2. Look for common factors of the numerator and denominator and cancel the common factor.
 - a. If there is a common factor, set the common factor equal to zero and solve. The solution is the location of the removable discontinuity (hole).
 - b. Substitute the solution in to the simplified rational function to find the y-coordinate of the removable discontinuity (hole).
3. Set the simplified denominator equal to zero and solve.
4. The solution is the location of the vertical asymptote.

▼ Examples: Find any vertical asymptotes or removable discontinuities and sketch the graph of the rational function near the vertical asymptote.

▼ Example 1: $f(x) = \frac{3}{x+2}$

▼ Example 2: $g(x) = \frac{x+1}{2x-6}$

▼ Example 3: $h(x) = \frac{3x^2-6x+2}{x-2}$

▼ Example 4: $R(x) = \frac{x^2-9}{x^2+x-2}$

▼ Example 5: $G(x) = \frac{x^2+2x-15}{x^2-25}$

▼ Example 6: $G(x) = \frac{x^4-16}{x^2+9}$

▼ Finding Horizontal and Oblique Asymptotes

Let f the rational function given by

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0}$$

The degree of the numerator (the highest power on x from the numerator) is n .
The degree of the denominator (the highest power on x from the denominator) is m .

1. If $n < m$, the x -axis, or $y = 0$ is the horizontal asymptote of the graph of f .

In words: If the degree of the numerator is less than the degree of the denominator then the horizontal asymptote is $y = 0$.

2. If $n = m$, the line $y = \frac{a_n}{b_m}$ is the horizontal asymptote of the graph of f .

In words: If the degree of the numerator is equal to the degree of the denominator then the horizontal asymptote is $y =$ the ratio of the leading coefficients.

3. If $n = m + 1$, the graph of f has no horizontal asymptote but does have an oblique asymptote. Use long division to find the oblique asymptote. The oblique asymptote is a linear equation of the form $y = mx + b$

In words: If the degree of the numerator is one more than the degree of the denominator then there is no horizontal asymptote but an oblique asymptote can be found using long division on the given rational function.

4. If $n \geq m + 2$ the graph of f has neither a horizontal asymptote nor an oblique asymptote. Use long division to determine the end behavior of the rational function.

In words: If the degree of the numerator is two or more larger than the degree of the denominator the graph has neither a horizontal asymptote nor an oblique asymptote.

▼ Examples: Find any horizontal or oblique asymptotes for the rational function.

▼ Example 1: $f(x) = \frac{3}{x+2}$

▼ Example 2: $g(x) = \frac{x+1}{2x-6}$

▼ Example 3: $h(x) = \frac{3x^2-6x+2}{x-2}$

▼ Example 4: $R(x) = \frac{x^2-9}{x^2+x-2}$

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▼ Example 6: $G(x) = \frac{x^4-16}{x^2+9}$

▼ Points where the graph crosses a Horizontal or Oblique Asymptote

To find the points where the graph crosses the horizontal asymptote or oblique asymptote set the function equal to the asymptote and solve.

1. Create an equation $f(x) =$ "the Horizontal Asymptote" or $f(x) =$ "the Oblique Asymptote"
2. Solve the equation. The solution is the x-coordinate of the point where the function crosses the asymptote
3. The y-coordinate of the point can be found by substituting the x-coordinate into the function or horizontal asymptote or the oblique asymptote.

▼ Examples: Find any points where the graph crosses the horizontal or oblique asymptote

▼ Example 1: $f(x) = \frac{3}{x+2}$

▼ Example 2: $g(x) = \frac{x+1}{2x-6}$

▼ Example 3: $h(x) = \frac{3x^2-6x+2}{x-2}$

▼ Example 4: $R(x) = \frac{x^2-9}{x^2+x-2}$

▼ Example 5: $G(x) = \frac{x^2+2x-15}{x^2-25}$

▼ Example 6: $G(x) = \frac{x^4-16}{x^2+9}$

▼ Finding the Domain of a Rational Function

▼ Finding Domain of Rational Functions

Since rational functions are in the form of a fraction, there is a restriction that the denominator is not zero. You can find the restricted values by creating an equation. The equation is the denominator equal to zero.

$$q(x) = 0$$

The solutions to this equation are restricted values and must be excluded from the domain.

▼ Examples: Find the domain of the rational function

▼ Example 1: $f(x) = \frac{3}{x+2}$

▼ Example 2: $g(x) = \frac{x+1}{2x-6}$

▼ Example 3: $h(x) = \frac{3x^2-6x+2}{x-2}$

▼ Example 4: $R(x) = \frac{x^2-9}{x^2+x-2}$

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▼ Example 6: $G(x) = \frac{x^4-16}{x^2+9}$

▼ Finding the Intercepts of a Rational Function

▼ Finding Intercepts Algebraically

To find an x-intercept: let $y = 0$ and solve for x . In function notation, let $f(x) = 0$ and solve for x . Be sure to check that the solution is in the domain of the rational function. You can avoid getting an extraneous x-intercept by canceling any common factors first.

To find an y-intercept: let $x = 0$ and solve for y . In function notation, find $f(0)$.

▼ Examples:

▼ Example 1: Find the x and y intercepts.

$$f(x) = \frac{3}{x+2}$$

▼ Example 2: Find the x and y intercepts.

$$R(x) = \frac{x^2-9}{x^2+x-2}$$

▼ Example 3: Find the x and y intercepts.

$$R(x) = \frac{x^2+2x-15}{x^2-25}$$

▼ Graphing Rational Function

▼ Example 1: Sketch a graph of the rational function

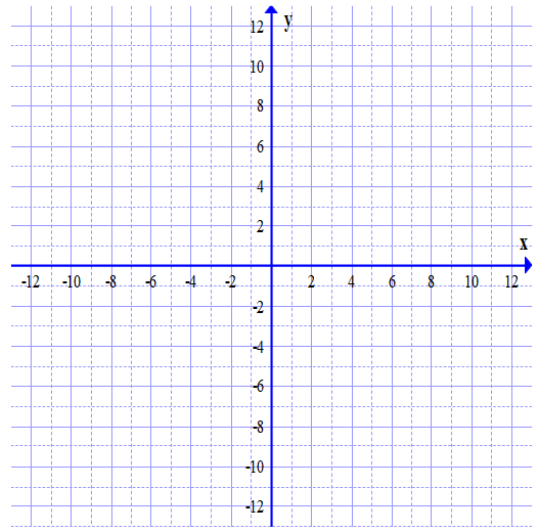
$$f(x) = \frac{2x}{x^2-9}$$

Find the domain.

Find any vertical asymptotes or holes in the graph

Find any horizontal or oblique asymptotes and any points where the function crosses these asymptotes

Find any x and y intercepts.



▼ Example 2: Sketch a graph of the rational function

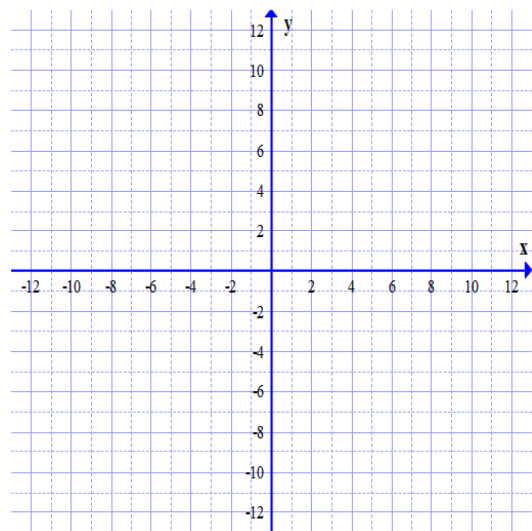
$$f(x) = \frac{3}{x+2}$$

Find the domain.

Find any vertical asymptotes or holes in the graph

Find any horizontal or oblique asymptotes and any points where the function crosses these asymptotes

Find any x and y intercepts.



▼ Example 3: Sketch a graph of the rational function

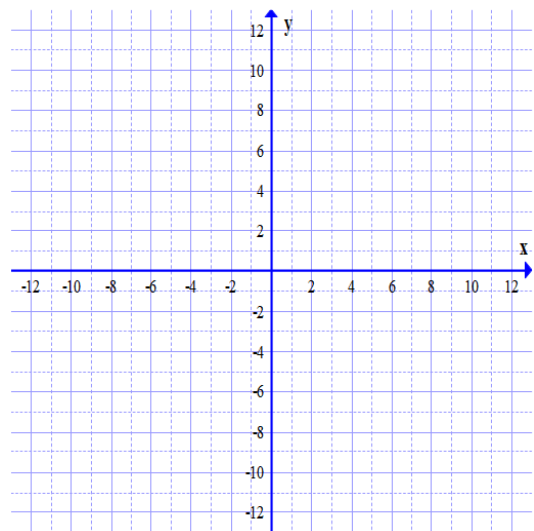
$$g(x) = \frac{x+1}{2x-6}$$

Find the domain.

Find any vertical asymptotes or holes in the graph

Find any horizontal or oblique asymptotes and any points where the function crosses these asymptotes

Find any x and y intercepts.



▼ Example 4: Sketch a graph of the rational function

$$R(x) = \frac{x^2 - 9}{x^2 + x - 2}$$

Find the domain.

Find any vertical asymptotes or holes in the graph

Find any horizontal or oblique asymptotes and any points where the function crosses these asymptotes

Find any x and y intercepts.

