9.1 Sequences

- Definition of a **Sequence**

  A **sequence** is a function whose domain is the set of positive integers.

- Representations of Sequences
  - The sequence is defined by nth term.
  - The sequence is defined by the first few terms of the sequence is list and the pattern is implied to continue
  - The sequence is defined recursively meaning that you can find a term of the sequence based on the previous term(s) of the sequence.

- Examples: Write the first six terms of sequences defined by the nth term
  - Example 1:

    $$\{a_n\} = \left\{\frac{n-1}{n}\right\}$$

  - Example 2:

    $$\{b_n\} = \left\{(-1)^{n+1}\left(\frac{2}{n}\right)\right\}$$

  - Example 3: Write the first six terms of the sequence.

    $$\{c_n\} = \begin{cases} n & \text{if n is even} \\ \frac{1}{n} & \text{if n is odd} \end{cases}$$

- Examples: Find the pattern and the nth term definition of sequences defined by first few terms
  - Example 1

    $$e, \frac{e^2}{2}, \frac{e^3}{3}, \frac{e^4}{4}, \ldots$$

  - Example 2

    $$1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$$

▼ Example 3

$1, 3, 5, 7, ...$

▼ Example 4

$1, 4, 9, 16, 25, ...$

▼ Example 5

$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, ...$

▼ Examples of sequences defined recursively

▼ Example 1: Write the first six terms of the sequence.

$s_1 = 1; s_n = ns_{n-1}$

▼ Example 2: Write the first six terms of the Fibonacci sequence

The original problem that Fibonacci investigated (in the year 1202) was about how fast rabbits could breed in ideal circumstances. Suppose a newly-born pair of rabbits, one male, one female, are put in a field. Rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits. Suppose that our rabbits never die and that the female always produces one new pair (one male, one female) every month from the second month on.

$u_1 = 1; u_2 = 1; u_n = u_{n-2} + u_{n-1}$

▼ Definition of **Factorial**

If $n \geq 0$ is an integer, the **factorial** symbol $n!$ is defined as follows:

- $0! = 1$
- $1! = 1$
- $n! = n(n-1)(n-2) \cdot ... \cdot 3 \cdot 2 \cdot 1$ if $n \geq 2$

▼ Example: Simplify the expression

$6!$

$\frac{6!}{4!(6-4)!}$

▼ Description of Summation Notation

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + ... + a_{n-1} + a_n$$

a_i is a sequence $a_1, a_2, a_3, ...$

The Greek symbol of Σ is used to indicate summation or adding the terms of the sequence

The expression is read "The sum of a_i from $i = 1$ to n"

▼ Examples using summation notation

▼ Example 1: Write out the sum.

$$\sum_{i=1}^{n} \frac{1}{i}$$

▼ Example 2: Write out the sum.

$$\sum_{i=1}^{n} i!$$

▼ Examples of writing the summation notation

▼ Example 1: Express the sum using summation notation

$1^2 + 2^2 + 3^2 + ... + 9^2$

▼ Example 2: Express the sum using summation notation

$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{2^{n-1}}$

- Property: Factor a constant multiple out of the sum

$$\begin{aligned}\sum_{i=1}^{n}(ca_i) &= ca_1 + ca_2 + ca_3 + ... + ca_n \\ &= c(a_1 + a_2 + a_3 + ... + a_n) \\ &= c\sum_{i=1}^{n} a_i\end{aligned}$$

- Property: Apply the sum to individual terms

$$\begin{gathered}\sum_{i=1}^{n} a_i \pm b_i \\ = (a_1 \pm b_1) + (a_2 \pm b_2) + (a_3 \pm b_3) + ... + (a_n \pm b_n) \\ = a_1 \pm b_1 + a_2 \pm b_2 + a_3 \pm b_3 + ... + a_n \pm b_n \\ = (a_1 + a_2 + a_3 + ... + a_n) \pm (b_1 + b_2 + b_3 + ... + b_n) \\ = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i\end{gathered}$$

- Property: Sum when the sequence doesn't start at 1

$$\sum_{i=j+1}^{n} a_i = \sum_{i=1}^{n} a_i - \sum_{i=1}^{j} a_i$$ where $0 < j < n$

- Summation Notation Formulas
    - Sum of a constant

      $$\sum_{i=1}^{n} c = cn$$

    - Sum of i

      $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

    - Sum of i^2

      $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

    - Sum of i^3

      $$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

- Examples applying summation notation formulas
  - Example 1:

    $$\sum_{i=1}^{6} 4i$$

  - Example 2:

    $$\sum_{i=1}^{26} i^3 + 2$$

  - Example 3:

    $$\sum_{i=1}^{10} i^2 - 3i + 4$$

  - Example 4:

    $$\sum_{i=3}^{8} 3i^2$$

  - Example 5:

    $$\sum_{i=1}^{n} \frac{i+1}{n^2}$$

  - Example 6:

    $$\sum_{i=1}^{n} \left(\frac{2(i-1)}{n}\right)^2 \cdot \frac{2}{n}$$