8.2 Systems of Linear Equations: Matrices

▼ Definition of a Matrix

A matrix is a rectangular array of numbers:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

▼ Definition of Augmented Matrix

The matrix used to represent a system of linear equations is called an **augmented matrix**.

▼ Example: Write a system of two equations as an augmented matrix

$$\begin{cases} 3x - 4y = -6 \\ 2x - 3y = -5 \end{cases}$$

$$\begin{cases} 2x - y + z = 0 \\ x + z - 1 = 0 \\ x + 2y - 8 = 0 \end{cases}$$

▼ Example: Write an augmented matrix as a system of equations

$$\left[\begin{array}{cc|c} 5 & 2 & 13 \\ -3 & 1 & -10 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3 & -1 & -1 & 7 \\ 2 & 0 & 2 & 8 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

- Row Operations

  1. Interchange any two rows
  2. Replace a row by a nonzero multiple of that row
  3. Replace a row by the sum of that row and a constant nonzero multiple of some other row.

- Definition of Row Echelon Form

  A matrix is in row echelon form when the following conditions are met:

  1. The entry in row 1, column 1 is a 1. and only 0's appear below it.
  2. The first nonzero entry in each row after the first row is a 1, only 0's appear below it and the 1 appears to the right of the first nonzero entry in any row above.
  3. Any rows that contain all 0's to the left of the vertical bar appear at the bottom.

$$\left[\begin{array}{ccc|c} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 0 & f \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 0 & 0 \end{array}\right]$$

- Definition of Reduced Row Echelon Form

  In this form, row operations are used to obtain entries that are 0 above (as well as below) the leading 1 in a row.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right]$$

- Example: Write the matrix in Row Echelon Form and Reduced Row Echelon Form

$$\left[\begin{array}{cc|c} 1 & -2 & 2 \\ 3 & -5 & 9 \end{array}\right]$$

▼ Steps for Matrix Method of Solving a System of Linear Equations

Step 1:	Write the augmented matrix that represents the system.
Step 2:	Perform row operations that place the entry 1 in row 1, column 1.
Step 3:	Perform row operations that leave entry 1 in row 1 column 1 unchanged, while causing 0's to appear below it in column 1.
Step 4:	Perform row operations that place the entry 1 in row 2, column 2, but leave the entries in the columns to the left unchanged. If it is impossible to place a 1 in row 2, column 2, proceed to place a 1 in row 2, column 3. Once a 1 is in place, perform row operations to place 0's below it.
Step 5:	Now repeat Step 4, placing a 1 in the next row, but one column to the right. Continue until the bottom row or the vertical bar is reached.
Step 6:	The matrix that results is the row echelon form of the augmented matrix. Analyze the system of equations corresponding to it to solve the original system.

▼ Solve the System using Matrices

$$\begin{cases} x - 2y = 2 \\ 3x - 5y = 9 \end{cases}$$

- Solve the System using Matrices

$$\begin{cases} 2x + 2y = 6 \\ x + y + z = 1 \\ 3x + 4y - z = 13 \end{cases}$$

- Solve the System using Matrices

$$\begin{cases} 6x - y - z = 4 \\ -12x + 2y + 2z = -8 \\ 5x + y - z = 3 \end{cases}$$

- Solve the System using Matrices

$$\begin{cases} x + y + z = 6 \\ 2x - y - z = 3 \\ x + 2y + 2z = 0 \end{cases}$$