6.3 Exponential Functions

▼ Definition: Exponential Function

An **exponential function** is of the form

$$f(x) = b^x$$

where *b* (referred to as the base) is a positive real number (*b>0*) that is not equal to one. The domain of *f* is the set of all real numbers.

A transformation of the the exponential function is often use to model exponential functions

$$f(x) = C \cdot b^x$$

The base b is the growth factor and C is the initial value.

▼ The natural base e

▼ Exploration

Find the value of $\left(1+\frac{1}{n}\right)^n$ as n increases to infinity.

Natural Base

n	f(n)
1	2
10	2.59374246
100	2.704813829
1000	2.716923932
10000	2.718145927
100000	2.718268237
1000000	2.718280469

▼ Definition: *e*

The **number *e*** is defined as the number that the expression

$$\left(1+\frac{1}{n}\right)^n$$

approaches as n approaches infinity. In calculus this is expressed using limit notation as

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

▼ Comparing a linear function to an exponential function

A **linear function,** $f(x) = mx + b$**,** has a **constant average rate of change.**

$$b = f(0)$$

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

- difference of consecutive outputs is constant
- y-intercept is the initial value

An **exponential function,** $f(x) = C \cdot b^x$**,** has a **constant ratio of consecutive outputs**

$$C = f(0)$$

$$b = \frac{f(x+1)}{f(x)}$$

- ratio of consecutive outputs is constant
- y-intercept is the initial value

▼ Example: Determine whether the given function is linear exponential or neither. For those that are linear or exponential find the equation that models the data.

▼

Example 1

x	f(x)
-2	5/4
-1	5/2
0	5
1	10
2	20

▼

Example 2

x	f(x)
-2	1
-1	3
0	5
1	7
2	9

▼

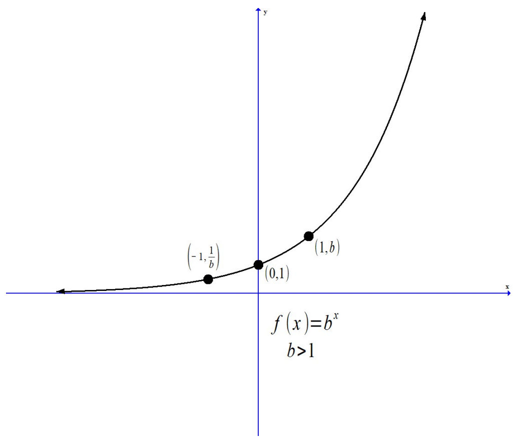
Example 3

x	f(x)
-1	2
0	4
1	7
2	11
3	16

▼ Properties

▼ Properties of an Exponential Function (b>1)

$$f(x) = b^x; b > 1$$



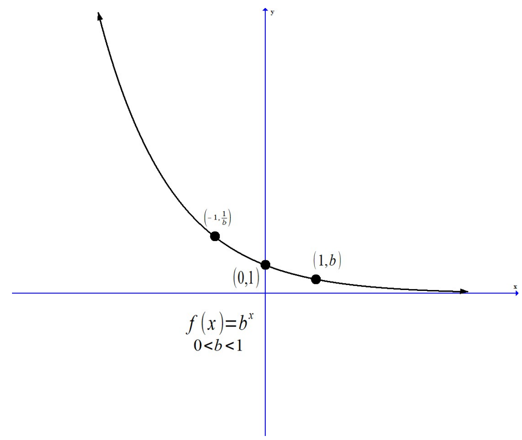
1. The domain is the set of all real numbers and the range is the set of all positive numbers.
2. There are no x-intercepts and the y-intercept is 1.
3. The x-axis (y=0) is a horizontal asymptote as x approaches negative infinity.

$$\lim_{x \to -\infty} f(x) = 0$$

4. *f(x)* is an increasing function and is one-to-one.
5. The graph of *f* is smooth and continuous, with no corners or gaps.

▼ Properties of an Exponential Function (0<b<1)

$$f(x) = b^x; 0 < b < 1)$$



1. The domain is the set of all real numbers and the range is the set of all positive numbers.
2. There are no x-intercepts and the y-intercept is 1.
3. The x-axis (y=0) is a horizontal asymptote as x approaches infinity.

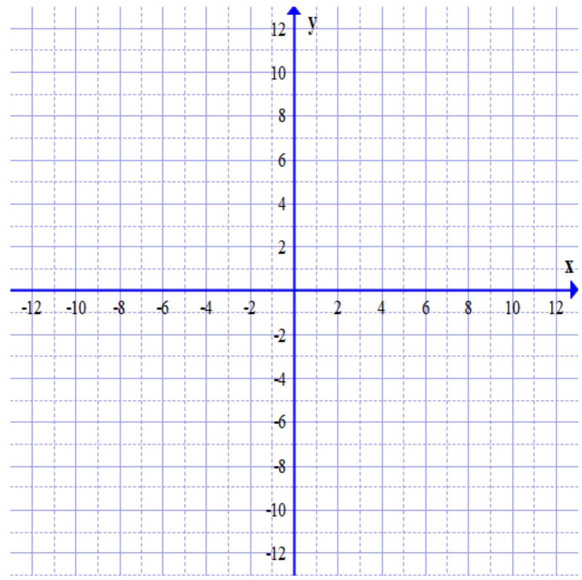
$$\lim_{x \to \infty} f(x) = 0$$

4. f(x) is an decreasing function and is one-to-one.
5. The graph of *f* is smooth and continuous, with no corners or gaps.

▼ Examples of Graphing Basic Exponential Functions

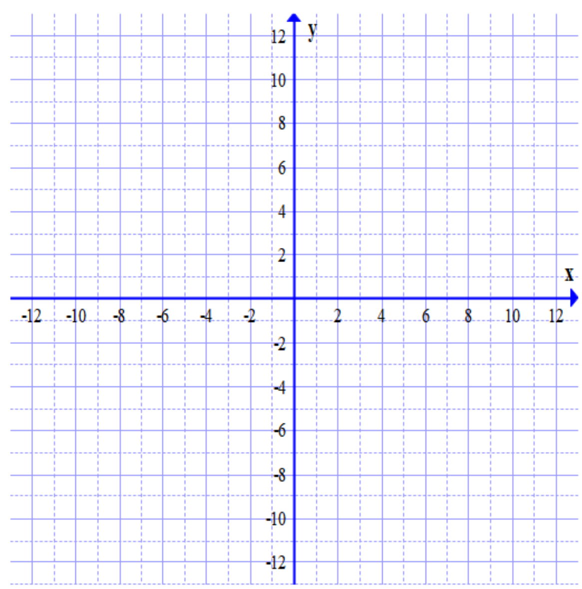
▼ Example 1: Graph $y = 2^x$ by plotting points

x	y	(x,y)



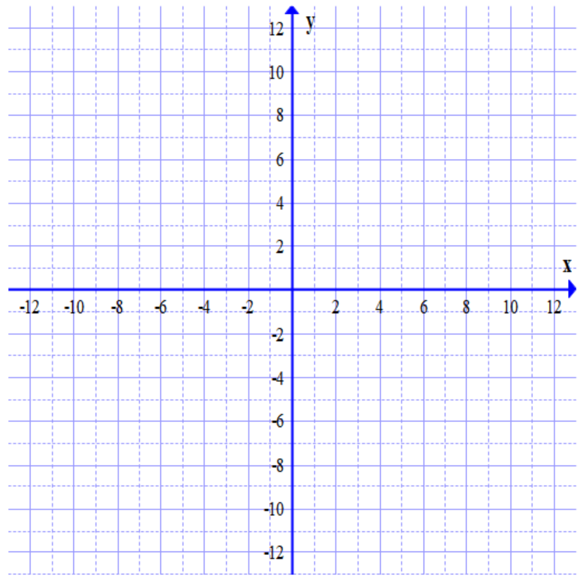
▼ Example 2: Graph $y = \left(\frac{1}{3}\right)^x$ by plotting points

x	y	(x,y)



- Example 3: Graph $y = e^x$ by plotting points

x	y	(x,y)



- Examples: Graphing Transformations
  - Example 1: Graph $f(x) = -2^{x-3} - 5$ by Transformations

Base Function: ________________
List Transformations:

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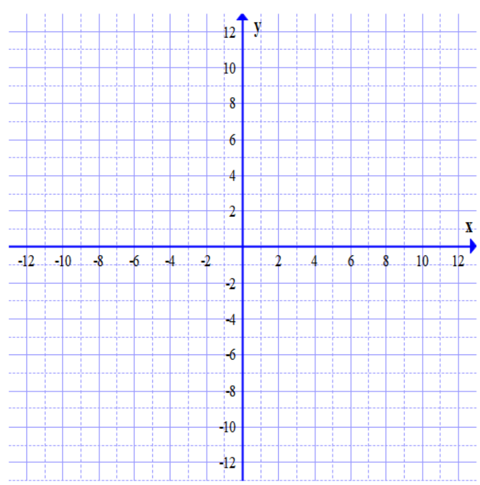

▼ Example 2: Graph $f(x) = -\left(\frac{1}{3}\right)^{x+5} + 2$ by Transformations

Base Function: ________________
List Transformations:

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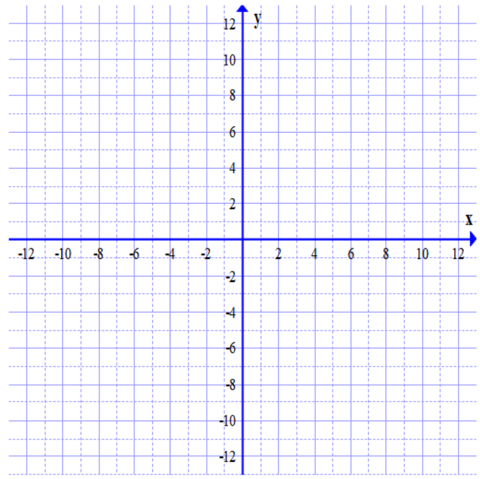

▼ Example 3: Graph $f(x) = -e^{-x}$ by Transformation

Base Function: ________________
List Transformations:

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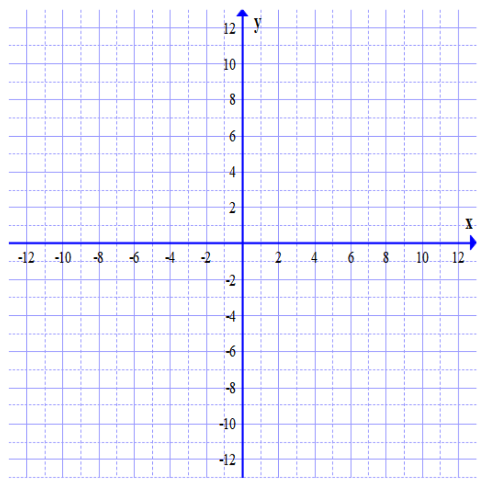

▼ One-to-one Property for Exponential Function (Relating the Bases)

If $b^u = b^v$ then $u = v$

This happens because exponential functions are one-to-one functions and by definition different x values have different y values and therefore if the y values are the same the x values are the same.

- Solving Exponential Equations: Relating the Bases or One-to-One Property
  - Method of Relating the Bases in Symbols

$$b^u = b^v$$
$$u = v$$

  - Method of Relating the Bases in Words
    - Requires the exponential equation to have the bases on both sides the same
    - When the bases are the same the exponents must be equal because of the one-to-one property of exponential functions.
  - Examples of Solving Exponential Equations by Relating the Bases
    - Example 1: Solve. $3^x = 81$
    - Example 2: Solve. $\sqrt{5} = 125^x$
    - Example 3: Solve. $\frac{1}{\sqrt[5]{216}} = 6^x$
    - Example 4: Solve. $e^x = \frac{1}{e^{13}}$

▼ Example 5: Solve. $5^{x-1} = \frac{1}{25}$

▼ Example 6: Solve. $3^{x+1} = 9^{5x}$

▼ Example 7: Solve. $\left(e^{x^2}\right)^5 = e^{180}$

▼ Example 8: Solve. $6^{x^3} = \frac{1296^x}{216^{-x^2}}$

▼ Example 9: Solve. $e^{x^2} = e^x \cdot e^{12}$

▼ Example 10: Solve. $e^{x^2} = \frac{e^8}{(e^x)^2}$

▼ Example: Applications of Exponential Equations

The average annual salary of an NBA player follows the exponential model

$$S(t) = 161.4(1.169)^t$$

where S(t) is the average annual salary in thousands of dollars and t is the number of years after 1980.

a. Find the average annual salary of an NBA players in 1980.

b. Find the average annual salary of an NBA players in 1990.

c. Find the average annual salary of an NBA players in 1998.