Properties of Rational Functions

Rational Functions are quotients of polynomial functions. A rational function can be expressed as $f(x)=\frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial functions where $q(x)\neq 0$.

The **domai**n of a rational function is the set of all real numbers except the x-values that make the denominator zero.

Finding the Domain of a Rational Function

Solve the equation $q(x)=0$. The solutions to this equation must be excluded from the domain.

Example: $f(x)=\frac{3x+1}{2x-6}$ Here $p(x)=3x+1$ and $q(x)=2x-6$ To find the domain we solve the equation $q(x)=0$.

$$
\begin{gathered}
2x-6=0\\
2x-6+6=0+6\\
2x=6\\
\frac{2x}{2}=\frac{6}{2}\\
x=3
\end{gathered}
$$

3 must be excluded from the domain of f. The domain of *f* is $(-\infty,3)\cup(3,\infty)$ or $\{x\mid x\neq 3\}$

Definition of a Vertical Asymptote

The line $x=a$ is a vertical asymptote of the graph of a function *f* if $f(x)$ increases or decreases without bound as *x* approaches *a*.

Locating Vertical Asymptotes

If $f(x)=\frac{p(x)}{q(x)}$ is a rational function in which $p(x)$ and $q(x)$ have no common factors and *a* is a zero of $q(x)$, the denominator, then $x=a$ is a vertical asymptote of the graph of *f*.

Example: $f(x)=\frac{3x+1}{2x-6}$ Notice first the the numerator and denominator have no common factor. To find a vertical asymptote you must find the zeros of the denominator $q(x)$. To find the zeros of the polynomial *q* you solve the equation $q(x)=0$.

$$
\begin{gathered}
2x-6=0\\
2x-6+6=0+6\\
2x=6\\
\frac{2x}{2}=\frac{6}{2}\\
x=3
\end{gathered}
$$

The vertical asymptotes is the vertical line $x=3$.

Example: $g(x)=\frac{x-1}{x^2-1}=\frac{x-1}{(x-1)(x+1)}=\frac{1}{x+1}$ In this example when we factor the denominator we notice that there is a common factor between the numerator and denominator. Because of that common factor there is only a vertical asymptotes at the line $x=-1$. At $x=1$, there is a hole in the graph.

Definition of a Horizontal Asymptote
The line $y=b$ is a horizontal asymptote of the graph of a function *f* if $f(x)$ approaches *b* as *x* increases without bound.

Definition of an Oblique or Slant Asymptote
The line $y=ax+b$ is a oblique asymptote of the graph of a function *f* if $f(x)$ approaches $ax+b$ as *x* increases without bound.

Locating Horizontal/Oblique Asymptotes
Let *f* the rational function given by

$$f(x)=\frac{p(x)}{q(x)}=\frac{a_n x^n+a_{n-1}x^{n-1}+\ldots+a_1 x+a_0}{b_m x^m+b_{m-1}x^{m-1}+\ldots+b_1 x+b_0}$$

The degree of the numerator is *n*. The degree of the denominator is *m*.
1. If $n<m$, the *x*-axis, or $y=0$, is the horizontal asymptote of the graph of *f*.

2. If $n=m$, the line $y=\frac{a_n}{b_m}$ is the horizontal asymptote of the graph of *f*.

3. If $n=m+1$, the graph of *f* has an oblique asymptote. Use long division to find the oblique asymptote.

4. If $n\geq m+2$, the graph of *f* has neither a horizontal asymptote nor an oblique asymptote. Use long division to determine the end behavior of the rational function.

Example: $f(x)=\frac{2x-1}{3x^2-3x-5}$ Here the degree of the numerator is 1 and the degree of the denominator is 2. Since the degree of the numerator is less than the degree of the denominator, there is a horizontal asymptote at $y=0$.

Example: $g(x)=\frac{5x^3-5x^2+4x-1}{2x^3+4x^2-4x+4}$ Here the degree of the numerator is 3 and the degree of the denominator is 3. Since they are the same, there is a horizontal asymptote at $y=\frac{5}{2}$

Example: $h(x)=\frac{2x^2+2x-1}{x+4}$ Here the degree of the numerator is 2 and the degree of the denominator is 1. Since the degree of the numerator one more than the degree of the denominator, there is an oblique asymptote. After doing long division we ignore the remainder and find the oblique asymptote to be $y=2x-6$.